Extra extension properties of equidimensional holomorphic mappings: results and open questions

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Abstract

Holomorphic (nondegenerate) mappings between complex manifolds of the same dimension are of special interest. For example, they appear as coverings of complex manifolds. At the same time they have very strong "extra" extension properties in compare with mappings in different dimensions. The aim of this paper is to put together the known results on this subject, give some perspective on the general strategy for future progress, prove some new results and formulate open questions.

1 Introduction.

1.1. Results. Most frequently a domain D which admits a group of holomorphic authomorphisms Γ acting on D properly discontinuously withought fixed points is itself homogeneous, *i.e.*, the group $Aut(D)$ of all biholomorphic authomorphisms of D is transitive. In that case the quotient $X = D/\Gamma$ is a compact homogeneous manifold. It turns out that much more can be said about the extension of locally biholomorphic mappings in the case when X is homogeneous and Kähler.

Recall that a domain (D, π) over a complex manifold M os called locally pseudoconvex (locally Stein) if for every point $a \in \overline{\pi(D)}$ there exists a neighborhood $V \ni a$ such that all connected components of $\pi^{-1}(V)$ are Stein.

Theorem 1. *Let X be a locally homogeneous complex manifold and let $f : \hat{D} \to X$ be a meromorphic mapping from a locally pseudoconvex domain $(\hat{D}, \pi)$ over a complex manifold M. Suppose that f is locally biholomorphic outside of its indeterminacy set. Then f is holomorphic (and therefore locally biholomorphic) everywhere.*

In, particular, if f is a locally biholomorphic mapping from a domain (D, π) over a Stein manifold M to a compact locally homogeneous Kähler manifold then f extends locally biholomorphically onto the envelpe of holomorphy $\hat{D}$ of D.

The proof is given in Theorem 2.4 and in the Remark 2.4 after the proof of Theorem 2.4 in Section 2.

Remark 1.1 The condition of compacity in this theorem, as in almost all results of this paper, can be relaxed to disk-convexity. See more about this in Section 6.

Remark 1.2 For more results of this type see Corollary 3.2, Corollary 4.1, Corollaries 5.1, 5.2, 5.3 and Proposition 5.1.

Let's now formulate an another type of result.

Theorem 2. *Let $p : X \to Y$ be a holomorphic fibration over a compact complex manifold with compact Kähler fibers. Let $S \subset Y$ be a closed subset such that $Y \setminus S$ is Stein. Then any meromorphic section of our fibration, defined in a neighborhood of S extends to a meromorphic section over the whole of Y.*

Remark 1.3 There is no assumption on how the Kähler metrics on fibers depend on the point on the base. Of course the total space X don't need to be Kähler and even locally Kähler.

The proof, which is based on results of extension of meromorphic mappins into non-Kähler manifolds is given in Section 4. More results are proved in Theorems 6.1 and 6.2.

1.2. Open Questions. A special accent in this paper is made to the open questions. This is done in order to present the authors vision of the future developpents in the area and in the hope to get interested new people to enter the subject.

Every section ends with few open questions relevant to thie section and the last Section 7 is entirely devoted to questions which concern the paper in general.

Remark 1.4 **1.** Mappings which decrease the dimension were studied in [St] and [Cha].
2. Mappings which increase the dimension do not have any extension properties in general, see example in [K1]. What one can expect at the best is explained in Section 3.
3. We do not discuss here an extremely extended topic of locally biholomorophic (or proper) mappings between domains in $\mathbb{C}^n$ and send the reader to the recent paper [NS] for the present state of art in this subject.

2 Coverings of Kähler Manifolds

2.1. General Kähler Manifolds. We start with the relatively well understandable case of Kähler manifolds. This case includes Stein manifolds, projective and quasiprojective manifolds. Results quoted in this section are directly applicable also to manifolds of class $\mathcal{C}$, *i.e.,* bimeromorphic to Kähler ones.

In [Iv1] the following theorem was proved:

Theorem 2.1 *Let X be a compact Kähler manifold. Then the following conditions on X are equivalent:*

i) for any domain D in a Stein manifold M, any holomorphic mapping $f : D \longrightarrow X$ extend to a holomorphic mapping $\hat{f} : \hat{D} \longrightarrow X$ from the envelope of holomorphy $\hat{D}$ of D into X.

ii) X doesn't contain rational curves, i.e., images of the Riemann sphere $\mathbb{CP}^1$ under a non-constant holomorphic mappings $\mathbb{CP}^1 \longrightarrow X$.

Remark 2.1 The condition on X to be compact is too restricitive. It can be replaced by the disk-convexity, see section 6.

As a corollary from this theorem we obtain a positive solution of the conjecture of Carlson et Harvey, see [CH]:

Corollary 2.1 *Let D be a domain in a Stein manifold M and let Γ be a subgroup of the group of holomorphic automorphisms of D acting on D properly discontinuously without fixed points. If $D/\Gamma = X$ is compact and Kähler then D is Stein itself.*

Indeed, such X cannot contain rational curves and therefore the covering map extends from D onto its envelope of holomorphy $\hat{D}$, which is Stein. Therefore the only possibility is $\hat{D} = D$ and therefore D is Stein itself. It can be viewed as certain generalization of the theorem of Siegel, [Sg]. In the theorem of Siegel D is supposed to be a bounded domain in $M = \mathbb{C}^n$ (as a result in this case X is, moreover, projective). If D is not necessarily bounded then X may not be algebraic (example: a non-algebraic torus as a quotient of $\mathbb{C}^n$ by a lattice).

Recall the following

Definition 2.1 *A meromorphic mapping f from a complex manifold D to a complex manifold X is an irreducible, locally irreducible analytic set $\Gamma_f \subset D \times X$ (graph of f) such that the natural projection $\pi_D : \Gamma_f \to D$ is proper and generically one to one.*

In that case there exists an analytic subset $I \subset D$ of codimension at least two such that $\Gamma_f \cap (D \setminus I) \times X$ is a graph of a holomorphic mapping (still denoted as f). This can be taken as a definition of a meromorphic mapping. The minimal I satisfying this property is called the indeterminacy set of f.

In [Iv2] the following conjecture of Griffiths, see [Gf], was proved:

Theorem 2.2 *For any domain D in a Stein manifold any meromorphic mapping from D into a compact Kähler manifold X extends to a meromorphic map from the envelope of holomorphy $\hat{D}$ of D into X.*

Remark 2.2 Again, as in Theorem 2.1 one needs only disk-convexity from X for result to be still true.

In the same way as above one obtains the following

Corollary 2.2 *Let D be a domain in a complex manifold M (not necessarily Stein!) and Γ a subgroup of the group of holomorphic automorphisms of D acting on D properly discontinuously without fixed points.*

(a) If D/Γ is compact and Kähler, then D is locally pseudoconvex.

(b) If $D/\Gamma = X'$ is Zariski open in a compact Kähler X, then D itself is Zariski open in a pseudoconvex domain $\hat{D}$ of M.

If $D \subset \mathbb{C}^n$ (b) is a theorem of Mok-Wong, [MW].

2.2. Homogeneous Kähler Manifolds.

Definition 2.2 *Complex manifold X is called infinitesimally homogeneous if the global sections of its tangent bundle generate the tangent space at each point.*

One can prove, see Proposition 1.3 in [Hr], that for some natural N there exists a surjective endomorphism of holomorphic bundles $\sigma : X \times \mathbb{C}^N \to TX$. This property can be taken as a definition of infinitesimally homogeneous manifold. All parallelizable manifolds are inf. hom., as well as all Stein manifolds and all complex homogeneous spaces under a real Lee group. Every Riemann domain (D, π) over an infinitesimally homogeneous manifold is infinitesimally homogeneous itself.

Using the morphism σ and some riemannian metric on X one can define, as in [Hr] a *boundary distance* function d_D on D. Ruffly speaking $d_D(z)$ for $z \in D$ is the supremum of the radii of balls B with centers in $\pi(z)$ such that π is injective over B. The principal result wee need from [Hr] is contained in Theorem 2.1. It can be stated as follows:

Theorem 2.3 *If (D, π) is a locally pseudoconvex domain with finite fibers over an infinitesimally homogeneous complex manifold. Then the function $-\log d_D(z))$ is plurisubharmonic.*

Note that no further assumptions on X (like compactness or Kählerness are needed).

We shall need also the Hironaka Resolution Singularities Theorem. We shall use the so called embedded resolution of singularities, see [Hi1], [BM]. Let us recall the notion of the sequence of blowings up over a complex manifold D. Take a smooth closed submanifold $l_0 \subset D_0 := D$ of codimension at least two. Denote by $\pi_1 : D_1 \longrightarrow D_0$ the blowing up of D_0 along l_0. Call this: *a blowing up of D_0 along the closed center l_0*. The exceptional divisor $\pi^{-1}(l_0)$ of this blowing up we denote by E_1.

We can repeat this procedure, taking a smooth closed submanifold $l_1 \subset E_1$ of codimension at least two in D_1 and produce D_2 and so on.

Definition 2.3 *A finite sequence $\{\pi^j\}_{j=1}^N$ of such blowings up we call a sequence of blowings up over $l_0 \in D$, or a regular modification over l_0.*

By $\{l_j\}_{j=0}^{N-1}$ we denote the corresponding centers and by $\{E_j\}_{j=1}^N$ the exceptional divisors. We put $\pi = \pi_1 \circ ... \circ \pi_N$, E denotes the exceptional divisor of π, i.e. $E = \pi_N^{-1}(l_{N-1} \cup ... \cup (\pi_1 \circ ... \circ \pi_N)^{-1}(l_0))$.

Let $f : D \to X$ be a meromorphic mapping into a manifold X. Denote by I the set of points of indeterminacy of f, *i.e.*, f is holomorphic on $D \setminus I$ and for every point $a \in D$ f is not holomorphic in any neighborhood of a.

Theorem. *Let $f : D \to X$ be a meromorphic map between complex manifolds D and X. Then there exists a regular modification $\pi : D_N \to D$ such that $f \circ \pi : D_N \to X$ is holomorphic.*

See [Hi1]. For the proof we refer also to [BM].

Theorem 2.4 *Let X be a compact infinitesimally homogeneous Kähler manifold. Then every locally biholomorphic mapping $f : D \to X$ from a domain D over a Stein manifold into X extends to a locally biholomorphic mapping $\hat{f} : \hat{D} \to X$ of the envelope of holomorphy $\hat{D}$ of D into X.*

Proof. Let $\hat{f} : \hat{D} \to X$ be the meromorphic extension of f. Denote by I the set of points of indeterminacy of $\hat{f}$. Then $\hat{f}|_{\hat{D}\setminus I}$ is locally biholomorphic and we can consider the pair $(\hat{D} \setminus I, \hat{f}|_{\hat{D}\setminus I})$ as a Riemann domain over X.

This domain may not be locally pseudoconvex only at points of I. But then its domain of existence $\tilde{D}$ over X contains some part of the exceptional divisor E of the desingularization of $\hat{f}$. The union of this part of E with $\hat{D} \setminus I$ is actually $\tilde{D}$ and the extension of $\hat{f}_{\hat{D}\setminus I}$ to $\tilde{D}$ we denote as $\tilde{f}$. We consider $(\tilde{D}, \tilde{f})$ as a (locally pseudoconvex) Riemann domain over X.

Suppose $\tilde{D} \setminus E$ is not empty. Then it is easy to construct a sequence of analytic discs Δ_k in $\hat{D} \setminus I$ and then in $\tilde{D}$ such that the boundaries of Δ_k stay in a compact part of $\tilde{D}$, but Δ_k converge to a disc plus some number of rational curves on $E \setminus \tilde{D}$. But this is clearly forbidden by the plurisubharmonicity of $-\log d_{\tilde{D}}$.

Therefore $(\tilde{D}, \tilde{f})$ as a domain over X coincides with $(\hat{D}_N, \hat{f}_N)$ - desingularization of $\hat{f}$. But then $-\log d_{\tilde{D}}$ should be constant on all fibers of our modification, because we can take as $\tilde{D}$ any locally pseudoconvex neighborhoods of these fibers. This is impossible unless these fibers are poins. That means that $\hat{f}_N = \hat{f}$ and therefore $\hat{f}$ is holomorphic on $\hat{D}$.

□

Remark 2.3 Theorem 2.2, Corollary 2.2 and Theorem 2.4 hold obviously true for manifolds of class $\mathcal{C}$, *i.e.,* for manifolds that are bimeromorphic to compact Kähler manifolds.

Remark 2.4 It is clear from the proof of Theorem 2.4 that the condition on X to be compact can be relaxed. In fact disk-convexity is sufficient, see Section 6. Kählerness of X was used also only once when we extended f onto the envelope of meromorphy. Therefore the Theorem 1 from the Introduction is also proved.

2.3. Open Questions. Let X be a compact Kähler surface and let $f : B^* \to X$ be a locally biholomorphic mapping of the punctured ball $B^* = B \setminus \{0\}$ into X. Then f extends meromorphically onto the whole ball B. The full image by the extension $\hat{f}$ of the origin denote by $E := \hat{f}[0]$.

Question 2.1 Prove that E is an exceptional curve in X.

Question 2.2 What can be said about $\hat{f}[I]$ in the conditions of Theorem 2.4?

3 Mappings into non-Kähler Manifolds

3.1. The Strategy. We start from the following remark. Let X be a compact complex manifold. Then due to the result of Gauduchon, see [Ga], X admits a Hermitian metric h such that its associated form ω_h satisfies $dd^c\omega_h^k = 0$, where $(k+1)$ is the complex dimension of X.

In fact we shall need a property which is easier to prove:

Every compact complex manifold of dimension $k+1$ carries a strictly positive (k,k)-form $\Omega^{k,k}$ with $dd^c\Omega^{k,k} = 0$.

Indeed: either a compact complex manifold carries a dd^c-closed strictly positive (k,k)-form or it carries a bidimension $(k+1,k+1)$-current T with $dd^cT \geqslant 0$ but $\not\equiv 0$. In the case of $\mathsf{dim}\, X = k+1$ such current is nothing but a nonconstant plurisubharmonic function, which doesn't exists on compact X.

Let us introduce the class $\mathcal{G}_k$ of normal complex spaces, carrying a nondegenerate positive dd^c-closed strictly positive (k,k)-forms. Note that the sequence $\{\mathcal{G}_k\}$ is rather exhaustive: $\mathcal{G}_k$ contains all compact complex manifolds of dimension $k+1$.

Introduce furthermore the class of normal complex spaces $\mathcal{P}_k^-$ which carry a strictly positive (k,k)-form $\Omega^{k,k}$ with $dd^c\Omega^{k,k} \leqslant 0$. Note that $\mathcal{P}_k^- \supset \mathcal{G}_k$. But Hopf three-fold $X^3 = \mathbb{C}^3 \setminus \{0\}/(z \sim 2z)$ belongs to $\mathcal{P}_1^-$ and not to $\mathcal{G}_1$, see remark below.

Consider the Hartogs figure

$$H_n^k(r) := \left[\Delta^n(1-r) \times \Delta^k\right] \cup \left[\Delta^n \times A^k(r,1)\right] \subset \mathbb{C}^{n+k}. \tag{1}$$

Here $\Delta^n(r)$ stands for the n-dimensional polydisk of radius r and $A^k(r,1) = \Delta^k(1) \setminus \bar{\Delta}^k(r)$ for the k-dimensional annulus (or shell). In (1) one should think about $0 < r < 1$ as being very close to 1.

General Conjecture. *Meromorphic mappings from $H_n^k(r)$ to compact (disk-convex) manifolds of class $\mathcal{P}_k^-$ should extend onto $\Delta^{n+k} \setminus A$ where A is of Haussdorf $(2n-1)$-dimensional measure zero. If the image manifold is from class $\mathcal{G}_k$ then $A \neq \varnothing$ should imply very restrictive conditions on the topology and complex structure of X (see results below).*

3.2. Mappings into Manifolds of Class $\mathcal{G}_1$. Let A be a subset of Δ^{n+1} of Hausdorff $(2n-1)$-dimensional measure zero. Take a point $a \in A$ and a complex two-dimensional plane $P \ni a$ such that $P \cap A$ is of zero length. A sphere $\mathbb{S}^3 = \{x \in P : \|x-a\| = \varepsilon\}$ with ε small will be called a "transversal sphere" if in addition $\mathbb{S}^3 \cap A = \varnothing$.

Theorem 3.1 *Let $f : H_n^1(r) \to X$ be a meromorphic map into a compact complex manifold X, which admits a Hermitian metric h, such that the associated $(1,1)$-form ω_h is dd^c-closed (i.e., $X \in \mathcal{G}_1$). Then f extends to a meromorphic map $\hat{f} : \Delta^{n+1} \setminus A \to X$, where A is a complete $(n-1)$-polar, closed subset of Δ^{n+1} of Hausdorff $(2n-1)$-dimensional measure zero. Moreover, if A is the minimal closed subset such that f extends onto $\Delta^{n+1} \setminus A$ and $A \neq \varnothing$, then for every transversal sphere $\mathbb{S}^3 \subset \Delta^{n+1} \setminus A$ its image $f(\mathbb{S}^3)$ is not homologous to zero in X.*

Remark 3.1 **1.** A (two-dimensional) *spherical shell* in a complex manifold X is the image Σ of the standard sphere $\mathbb{S}^3 \subset \mathbb{C}^2$ under a holomorphic map of some neighborhood of $\mathbb{S}^3$ into X such that Σ is not homologous to zero in X. Theorem 3.1 states that if the singularity set A of our map f is non-empty, then X contains spherical shells.

A good example to think about is a Hopf surface $H^2 = \mathbb{C}^2 \setminus \{0\}/(z \sim 2z)$ with the pluriclosed metric form $\omega = \frac{i}{2}\frac{dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2}{\|z\|^2}$.

2. Consider now a Hopf three-fold $H^3 = (\mathbb{C}^3 \setminus \{0\})/(z \sim 2z)$. The analogous metric form $\omega = \frac{i}{2}\frac{dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2 + dz_3 \wedge d\bar{z}_3}{\|z\|^2}$ is not longer pluriclosed but only plurinegative (i.e. $dd^c\omega \leqslant 0$). Moreover, if we consider ω as a bidimension $(2,2)$ current, then it will provide

us a natural obstruction for the existence of a pluriclosed metric form on H^3. That means that $H^3 \in \mathcal{P}_1^- \setminus \mathcal{G}_1$. The natural projection $f : \mathbb{C}^3 \setminus \{0\} \to H^3$ has singularity of codimension three and H^3 doesn't contains spherical shells of dimension two (but contain a spherical shell of dimension three). [Iv4] contains extension theorem for mappings into manifolds from class $\mathcal{P}_1^-$ also.

Later on in this paper we shall need one corollary from the Theorem 3.1. A real two-form ω on a complex manifold X is said to "tame" the complex structure J if for any non-zero tangent vector $v \in TX$ we have $\omega(v, Jv) > 0$. This is equivalent to the property that the $(1,1)$-component $\omega^{1,1}$ of ω is strictly positive. Complex manifolds admitting a *closed* form, which tames the complex structure, are of special interest. The class of such manifolds contains all Kähler manifolds. On the other hand, such metric forms are dd^c-closed. Indeed, if $\omega = \omega^{2,0} + \omega^{1,1} + \bar{\omega}^{2,0}$ and $d\omega = 0$, then $\partial\omega^{1,1} = -\bar{\partial}\omega^{2,0}$. Therefore $dd^c\omega^{1,1} = 2i\partial\bar{\partial}\omega^{1,1} = 0$. So the Theorem 3.1 applies to meromorphic mappings into such manifolds. In fact, the technique of the proof gives more:

Corollary 3.1 *Suppose that a compact complex manifold X admits a strictly positive $(1,1)$-form, which is the $(1,1)$-component of a closed form. Then every meromorphic map $f : H_n^1(r) \to X$ extends onto Δ^{n+1}.*

Remark 3.2 *1.* In particular, all results of Section 2 remain valid for such manifolds.
2. Theorem 3.1 stays valid for meromorphic mappings from all $H_n^k(r)$ for all $k \geqslant 1$. But it should be noted that in general extendibility of meromorphic mappings into some complex manifold X from $H_n^k(r)$ doesn't imply extendibility of meromorphic mappings onto this X neither from $H_{n+1}^k(r)$ no from $H_n^{k+1}(r)$ (for holomorphic mappings this is true), see example in [Iv6].

3.3. Class $\mathcal{G}_2$ and Dimension 3. The following result was proved in [IS].

Theorem 3.2 *Let X be a compact complex space of dimension 3 (more generally one can suppose that X is of any dimension but carries a positive dd^c-closed $(2,2)$-form). Then every meromorphic map $f : H_1^2(r) \to X$ extends meromorphically onto $\Delta^3 \setminus A$, where A is a zero-dimensional complete pluripolar set. If A is non-empty then for every ball B with center $a \in A$ such that $\partial B \cap A = \varnothing$, $f(\partial B)$ is not homologous to zero in X, i.e., $f(\partial B)$ is a spherical shell (of dimension 3) in X.*

Remark 3.3 Spherical shell of dimension k in complex manifold (space) X is an image Σ of the unit sphere $\mathbb{S}^{2k-1} \subset \mathbb{C}^k$ under a meromorphic map h from a neighborhood of $\mathbb{S}^{2k-1}$ into X such that $\Sigma = h(\mathbb{S}^{2k-1})$ is not homologous to zero in X.

Results of such type have interesting applications to coverings of compact complex manifolds as we shall see in the next sections. From this theorem immediately follows that if the covering manifold $\tilde{V}$ of a 3-dimensional manifold V is itself a subdomain in some compact complex manifold Y then he boundary of $\tilde{V}$ cannot have concave points.

Let's give one more precise statement. Recall that a complex manifold is called affine if it admits an atlas with affine transition functions. In that case its universal covering is a domain over $\mathbb{C}^n$.

Corollary 3.2 *Let V be a compact affine 3-fold and let $(\tilde{V}, \pi)$ be its universal covering considered as a domain over $\mathbb{C}^3$ with locally biholomorphic projection π. Then if $(\tilde{V}, \pi)$ is pseudoconcave at some boundary point then V contains a spherical shell (of dimension* 3*).*

Indeed, by the Theorem 3.2 the covering map $p : \tilde{V} \to V$ can be extended to a neighborhood of a pseudoconcave boundary point, say a, minus a zero dimensional set A. But this cannot happen unless $\tilde{V} = V \cup A$ in a neighborhood of a. Therefore spheres around a project to shells in V by the Theorem 3.2.

Remark 3.4 Of course, an analogous result can be formulated for affine surfaces: either the universal cover of an affine surface V is Stein or V contains a spherical shell (of dimension two).

3.4. Open Questions.

Question 3.1 We conjecture that the analogous result should hold for meromorphic mappings in all dimensions. I.e. from $H_n^k(r)$ to compact manifolds (and spaces) in the classes $\mathcal{P}_k^-$ and $\mathcal{G}_k$. In particular, Theorem 3.2 should be true for meromorphic mappings between equidimensional manifolds in all dimensions.

The main difficulty lies in the fact that it is impossible in general to make the reductions (a)–(c) of §1 from [IS]. (Note that reductions (d)–(e) can be achieved in all dimensions.)

Question 3.2 One can start proving the general conjecture (as in Question 3.1) by considering extension from $H_2^2(r)$ to a manifold of class $\mathcal{G}_2$.

Question 3.3 An analog of Corollary 3.2. in all dimensions seems to be an easier problem then the Question 3.1 in its whole generality.

It would be instructive to consult the paper [BK] in this regard.

It is likely that one can say more about the singularity set A of the extended mapping in Theorems 3.1 and 3.2.

Question 3.4 Let X is a compact complex manifold carrying a plurinegative metric form, and let $f : \Delta^3 \setminus S \to X$ is a meromorphic mapping. Suppose that A is a minimal closed subset of Δ^3 such that f extends onto $\Delta^3 \setminus A$. Prove that each connected component of A is a complex curve.

For general X without special metrics the answer could be negative, see examples in the last section of [Iv4].

4 Application to Kähler Fibrations

4.1. Extension of Meromorphic Sections. We start with the proof of the Theorem 2 from the Introduction, which answers a question posed to the author by T. Ohsawa.

Proof. *Step 1. Every point $y \in Y$ has a neighborhood U such that $X_U = p^{-1}(U)$ - the union of fibers over U, possesses a Hermitian metric such that its Kähler form ω_U is a (1,1)-component of a closed form.*

To see this take a coordinate neighborhood $U \ni y$ such that X_U is diffeomorphic to $U \times X_y$. Let $p_1 : U \times X_y$ be the projection onto the second factor. Let ω_y be a Kähler form on X_y. Consider the following 1-form on X_U: $\omega_U = p^* dd^c|z|^2 + p_1^*\omega_y$, where z is the vector of local coordinates on U. ω_U is d-closed. Its $(1,1)$-component is positive for U small enough, since ω_y is positive on X_y.

Let ρ be a strictly plurisubharmonic Morse exhaustion function on the Stein manifold $W := Y \setminus S$. Set $W_t = \{y \in W : \rho(y) > t\}$. Given a meromorphic section v on the neighborhood of S. Then v is defined on some W_t. The set T of t such that v meromorphically extends onto W_t is non-empty and close.

Step 2. T is open.

Let $t \in T$, then v is well defined and meromorphic on W_t. Set $S_t = \{y \in W : \rho(y) = t\}$. Fix a point $y_0 \in S_t$. Take a neighborhood U of y_0 and form ω_U as in the *Step 1*. If y_0 is a regular point of S_t then there exists a Hartogs figure $H \subset W_t$ such that the corresponding polydisk $D \ni y_0$. By Corollary 3.1 the meromorphic mapping $\mathsf{v} : H \to D \times X_{y_0}$ can be meromorphically extended to D and we are done.

If y_0 is a critical point of S_t then we use the result of Eliashberg, see also Lemma 2.1 from [FS]. By this description of critical points of strictly plurisubharmonic Morse functions we can suppose that y_0 lies on a totally real disk B in D and $W_t \supset D \setminus B$. But then the argument remains the same, because in this case we can also find an appropriate Hartogs figure in W_t with corresponding polydisk containing y_0.

Therefore v extends to W_t for all t and the Theorem is proved.

$\square$

Remark 4.1 Note that dealing with Kähler fibrations we were forced to use Corollary 3.1 which concerns non-Kähler situation.

4.2. Non-Kähler deformations of Kähler manifolds. Recall that a complwx deformation of a compact complex manifold X is a complex manifold $\mathcal{X}$ together with a proper surjective holomorphic map $\pi : \mathcal{X} \to \Delta$ od rank one with connected fibers and such that the fiber X_0 over zero is biholomorphic to X. From [Hi2] one knows that if X_0 is Kähler this doesn't implies that the neighboring fibers X_t are Kähler. But the Step 1 in the proof of the Theorem 4.1 tells us that for $t \sim 0$ the fiber X_t admits a Hermitian metric such that its associated form is a $(1,1)$-component of a closed form. Therefore Corollary 3.1 applies to X_t.

Let's give the formal statement. We say that a complex manifold X possesses a meromorphic extension property if for every domain D in Stein manifold every meromorphic mapping $f : D \to X$ meromorphically extends onto the envelope of holomorphy of D.

Corollary 4.1 *Let X_t be a complex deformation of a compact Kähler manifold X_0. Then for $t \sim 0$ X_t possesses a meromorphic extension property.*

4.3. Open Questions.

Question 4.1 Suppose all X_t for $t \neq 0$ possed meromorphic extension property (as, for example, Kähler manifolds). Does X_0 possesses it as well?

Question 4.2 If X_0 possesses a mer. ext. prop. does X_t possesses it for t close to zero?

5 Coverings of non-Kähler Manifolds

5.1. General facts. To stay within reasonable generality we shall restrict ourselves here with subdomains of $\mathbb{CP}^n$ covering compact complex manifolds (this includes also subdomains of $\mathbb{C}^n \subset \mathbb{CP}^n$). Hovewer many statements have an obvious meaning (reformulation) in the case of domains in general complex manifolds.

Locally pseudoconvex domains in (and over) both $\mathbb{C}^n$ and $\mathbb{CP}^n$ are Stein (with one exception - $\mathbb{CP}^n$ itself), see [Ok, T]. They can cover both Kähler and non-Kähler manifolds. But Theorem 2.2 imply that:

Corollary 5.1 *If a subdomain $D \subset \mathbb{CP}^n$ covers a compact Kähler manifold V then D is Stein, unless $V = \mathbb{CP}^n$.*

An example of Stein domain covering a non-Kähler compact manifold is any Inoue surface with $b_2 = 0$. Their universal covering is $\mathbb{C} \times H$, where H is the upper half-plane of $\mathbb{C}$.

5.2. Coverings by domains from $\mathbb{CP}^2$. Since every compact complex surface admits a dd^c-closed metric form, the Theorem 3.1 applies and we get:

Corollary 5.2 *If a subdomain $D \subset \mathbb{CP}^2$ covers a compact complex surface X then either D is Stein, or $\mathbb{CP}^2$ itself, or X contains a spherical shell.*

In $\mathbb{CP}^3$ we have an analogous corollary from Theorem 3.2. Recall that a domain $D \subset \mathbb{CP}^n$ is q-convex if it admits an exhaustion function such that its Levi form has at least $n-q+1$ strictly positive eigenvalues at each point outside of some compact subset of D.

Corollary 5.3 *If $D \subset \mathbb{CP}^3$ covers a compact complex 3-fold then either D is 2-convex, or $D = \mathbb{CP}^3$, or V contains a (three dimensional) spherical shell.*

5.3. Coverings by "large" domains from $\mathbb{CP}^3$. A domain $D \subset \mathbb{CP}^n$ is said to be "large" if its complement $\Lambda := \mathbb{CP}^n \setminus D$ is "small" in some sense. Different authors give different sense to the notion of being "small", see [K2, L] and therefore we shall reserve ourselves from giving a general definition.

Let's start from the remark that if $\Lambda \neq \varnothing$ then its Haussdorf n-dimensional (resp. $n-1$-dimensional) measure is non-zero if n is even (resp. odd). For example in $\mathbb{CP}^2$ and in $\mathbb{CP}^3$ this condition is the same: $h_2(\Lambda) > 0$, see [L]. Both cases is easy to realise by examples. We have the following

Proposition 5.1 *Suppose a domain $D \subset \mathbb{CP}^3$ covers a compact complex threefold X.*
Case 1. If the complement $\Lambda = \mathbb{CP}^3 \setminus D$ is locally a finite union of two-dimensional submanifolds, then Λ is a union of finitely many lines.
Case 2. If the complement $\Lambda = \mathbb{CP}^3 \setminus D$ is locally a finite union of three-dimensional submanifolds, then Λ is foliated by lines.

Proof. Take a point p on the limit set Λ and find a point $q \in D$ and a sequence of authomorphisms $\gamma_n \subset \Gamma$ such that $\gamma_n(q) \to p$. Here Γ is a subgroup of $Aut(D)$ such that $D/\Gamma = X$. Due tp the Haussdorf dimension condition on Λ there exists a line $l \ni q$ such that $l \cap \Lambda = \varnothing$. Then $\gamma_n(l)$ will converge to a line in Λ passing through p.

□

In [K2] an example of Λ of dimension 3 is constructed.

5.4. Open Questions.

Question 5.1 Prove an analog of the Case 1 of Theorem 5.1 assuming only that $h_2(\Lambda)$ is finite.

In that case the components of Λ could be lines and points.

Question 5.2 Suppose that the complement $\Lambda = \mathbb{CP}^3 \setminus D$ is locally a union of four-dimensional submanifolds. Are all components of Λ necessariry either complex hypersurfaces or CR-manifolds of CR-dimension one? Or, one can have components which are not CR-submanifolds? Are those CR-submanifolds Levi-flat?

[BK] contains an example on pp. 82-83 where one component of Λ is a complex hyperplane, and another is a Levi-flat "perturbation" of a complex hyperplane.

6 Disk-Convexity of Complex Spaces

6.1. The notion of disk-convexity. All results, except that of Section 4, presented in this paper are valid for more general classes of complex manifolds and spaces then just compact ones. Compacity can be replaced by much less restrictive condition, namely by disk-convexity.

Definition 6.1 *(a) Complex space X is called disk-convex if for every compact $K \subset X$ there is another compact $\hat{K}$ such that if for any holomorphic map $h : \bar{\Delta} \to X$ with $h(\partial\Delta) \subset K$ one has $h(\Delta) \subset \hat{K}$.*

(b) X is called disk-convex in dimension k if for every compact $K \subset X$ there is another compact $\hat{K}$ such that if for any meromorphic map $h : \bar{\Delta}^k \to X$ with $h(\partial\Delta^k) \subset K$ one has $h(\Delta^k) \subset \hat{K}$.

Remark 6.1 **1.** In all formulations of Section 2 "compact Kähler" can be replaced by "disk-convex Kähler". Neither original proofs no backgrounds use more then disk-convexity.

2. In formulations of Subsection 3.2 the same: "compact of class $\mathcal{G}_1$" can be replaced by "disk-convex of class $\mathcal{G}_1$". This was actually done in [Iv4], see Theorem 2.2. there.

3. Theorem 3.2 is valid for manifolds from $\mathcal{G}_2$ which are disk-convex in dimension 2.

6.2. k-convexity $\Longrightarrow$ disk-convexity in dimension k. Now let us compare the notion of disk-convexity with other convexities used in complex analysis. We shall see that our notion is the most weaker one (and this is its great advantage).

Definition 6.2 *A $\mathcal{C}^2$-smooth real function ρ on X is called k-convex if for any local chart $j : V \longrightarrow \tilde{V} \subset \Delta^N$ there exists a real $\mathcal{C}^2$-function $\tilde{\rho}$ on Δ^N such that $\tilde{\rho} \circ j = \rho$ and the Levi form of $\tilde{\rho}$ has at least $N - k + 1$ positive eigenvalues at each point of Δ^N.*

Definition 6.3 *Complex space X is called k-convex (in the sense of Grauert) if there exists a $\mathcal{C}^2$ exhaustion function $\rho : X \longrightarrow [0, \infty[$, which is k-convex outside some compact $K \Subset X$.*

We shall start with the following

Maximum Principle. *Let ρ be a k-convex function on the complex space X and A be a pure k-dimensional analytic subset of X. If for some point $p \in A$ $\rho(p) = \mathsf{sup}_{a\in A}\rho(a)$, then $\rho\mid_A \equiv const$.*

Proof. If there is a smooth point $p \in A^{\text{reg}}$ where $\rho \mid_A$ achieves its maximum, then conclusion is clear. Really, while the Levi form of $\rho_A := \rho \mid_A$ has at least one positive eigenvalue at p, one can find an analytically imbedded disk $\Delta \ni p$ such that the restriction $\rho \mid_\Delta$ is subharmonic. This implies that $\rho \mid_\Delta \equiv$ *const*. Further one can find a holomorphic coordinates $(z_1, ..., z_k) = (z_1; z^{'})$ in the neighborhood of p such that restriction ρ_D of ρ onto the every disk $D = \{(z_1; z^{'}) : z^{'} = 0\}$ is subharmonic and such that our original disk Δ is transversal to all such D. We conclude that $\rho \equiv$ *const* in the neighborhood of p. The rest is obvious.

Now consider the case when $p \in A^{\text{sing}}$ – the set of singular points of A. We shall be done if we shall prove that in the neighborhood of p there is another point $q \in A^{\text{reg}}$ such that $\rho(q) = \rho(p)$. Take a neighborhood V of p together with imbedding $j : V \longrightarrow \tilde{V} \subset \Delta^N$ of V as a closed analytic subset $\tilde{V}$ in the unit polydisk. Let also $j(p) = 0$. By $\tilde{A}$ let us denote $j(A \cap V)$ – an analytic subset of pure dimension k in Δ^N. Take some irreducible component B of $\tilde{A}$ passing through zero.

Lemma 6.1 *Let Π be a linear subspace of dimension $N - k + 1$ of $\mathbb{C}^N$. Then for a subspace which is a generic perturbation of Π (and is again denoted as Π) there exists an $\varepsilon > 0$ such that $\tilde{\Pi} \cap B \cap \Delta^N_\varepsilon$ is a complex curve.*

Proof. Blow up the origin in $\mathbb{C}^N$. Let $\mathbb{P}^{N-1}$ is an exceptional divisor and $\pi : \mathbb{C}^N \setminus \{0\} \to \mathbb{P}^{N_1}$ a natural projection. Denote by $\hat{B}$ and $\hat{\Pi}$ strict transforms of B and Π. Recall that $\pi^{-1}(\hat{B} \cap \mathbb{P}^{N-1}) \cup \{0\}$ is a tangent cone to B at zero. While $\hat{B} \cap \mathbb{P}^{N-1}$ is of dimension $k-1$ and $\hat{\Pi} \cap \mathbb{P}^{N-1}$ is a linear subspace of dimension $N - k$, then for a generic perturbation Π the intersection $\hat{\Pi} \cap \mathbb{P}^{N-1} \cap \hat{B}$ is zerodimensional.

The usual properties of tangent cone imply that $\Pi \cap B$ has the tangent cone at zero of dimension one. And this implies that for a small enough $\varepsilon > 0$ the intersection this in $\Pi \cap B \cap \Delta^n_\varepsilon$ is a curve.

Lemma is proved.

Let us finish the proof of the maximum principle. While the Levi form of $\tilde{\rho} := \rho \circ j$ has at least $N - k + 1$ positive eigenvalues at zero, one can find a linear subspace Π in $\mathbb{C}^n$

of dimension $N-k+1$ lying inside the positive cone of $\mathcal{L}_{\tilde{\rho}}(0)$. We can take instead of $\tilde{A}$ some of its irreducible component B passing through zero. After a small perturbation Π became transversal to B^{sing} still being in the positive cone. Thus $\Pi \cap B^{\mathsf{reg}} \cap \Delta_{\varepsilon}^{N} \neq \varnothing$ for all $\varepsilon > 0$ small enough and the same is true for the small perturbations of Π. Now our lemma provides us with a perturbation Π such that:
1) $\Pi \cap B \cap \Delta_{\varepsilon}^{N} =: C$ is a curve, passing through zero for some $\varepsilon > 0$;
2) Π lies in the positive cone of $\mathcal{L}_{\tilde{\rho}}(0)$;
3) $C \cap B^{\mathsf{reg}} \neq \varnothing$.

But this means that $\tilde{\rho} \mid_{C}$ is subharmonic. Having zero as maximum it is constant. Thus we have found smooth points where ρ takes its maximum.

q.e.d.

Theorem 6.1 *k-convexity $\Longrightarrow$ disk-convexity in dimension k.*

Proof. Let ρ be an exhaustion function on X, which is k-convex outside compact P. Put $a = \mathsf{sup}_{x\in K\cup P}\rho(x)$, and put $\hat{K} = \{x \in X : \rho(x) \leqslant a\}$. Let $h : \bar{\Delta}^{k} \to X$ be some meromorphic map with $h(\partial\Delta^{k}) \subset K$. Would $h(\bar{\Delta}^{k})$ be not contained in $\hat{K}$ then $h(\bar{\Delta}^{k})\setminus\hat{K}$ would be a nonempty pure k-dimensional analytic subset in $X \setminus \hat{K}$.

This clearly contradicts the maximum principle.

□

Remark 6.2 This Theorem answers the question which was posed to the Author by D. Barlet. It is well known that k-convexity is nearly weakest notion among convexities used in complex analysis.

6.3. Filling "holes" in Complex Surfaces. How fare can be a complex manifold or space be from being disk-convex? This seems to be a difficult question. Here we shall indicate an interesting particular case of being non-disk-convex. For the technical reasons we shall restrict ourselves to complex dimension two. $B^{*} = B \setminus \{0\}$ will stand for the punctured ball in $\mathbb{C}^{2}$.

Let X be a normal complex surface, *i.e.,* a normal complex space of complex dimension tao, which will be supposed to be reduced and countable at infinity. Following [AS] we give the following

Definition 6.4 *We say that X has a hole if there exists a meromorphic mapping f : $B^{*} \to X$ such that $\mathsf{lim}_{z\to 0} f(z) = \varnothing$.*

Remark 6.3 If X has a "hole" then it is certainly not disk-convex.

But this particular cause of non-disk-convexity can be repaired.

Theorem 6.2 *Let X be a complex surface. Then there exists a complex surface $\hat{X}$ and a meromorphic injection $i : X \to \hat{X}$ such that*

i) $i(X)$ is open and dense in $\hat{X}$;

ii) $\hat{X}$ has no holes.

Remark 6.4 This result was announced in [Iv5], here we shall give the sketch of the proof which crucially uses results of Grauert about complex equivalent relations, see [Gr1, Gr2]..

Proof. Let a "hole" $f : B^* \to X$ be given.If there is a curve $C \subset B^*$ contracted by f to a point $p \in X$ we can blow-up X at p and get a new surface and a new map which is not contracting C and which is still a "hole". Since, after shrinking B,there can be only finitely many contracted curves we can suppose without loss of generality that

$$f \text{ is not contracting any curves in } B^*.$$

On B^* we define the following equivalence relation $x\ y$ if $f(x) = f(y)$. This means that if one of these points, say y is an indeterminacy point of f then $f(x) \in f[y]$. If both x and y are points of indeterminacy then we require that $f[x] = f[y]$. This equivalence relation $R \subset B^* \times B^*$ is an analytic set in $B^* \times B^*$. This follows from the fact that f is a "hole". Really, one cannot have an accumulation point of R of the kind $(a, 0)$ of $(0, a)$ with $a \neq 0$. Moreover R is semiproper for the same reason. Therefore R extends to $B \times B$ it is a meromorphic equivalence relation there in the sense of [Gr2]. By the results of [Gr1, Gr2] the quotient $Q = B/R$ is a normal complex surface.

Now we can attach Q to X by $f|_R$ - quotient map and get a new normal surface with a "hole" filled in.

Using Zorn lemma one constructs a maximal extension $\hat{X}$ of X such that X is open and dense in $\hat{X}$ ($\hat{X}$ is not unique!). The "filling in" procedure above implies that this $\hat{X}$ should be Disk-convex.

□

6.4. Open Questions. One could try to improve the result of Theorem 6.1:

Question 6.1 Can every complex surface be imbedded as a subdomain into a disk-convex complex surface?

In some cases another notions of "disk-convexity" are needed:

i) A complex manifold X is said to be disk-convex if for any compact $K \subset X$ there exists a compact $\hat{K} \subset X$ such that for every Riemann surface with boundary $(R, \partial R)$ and every holomorphic mapping $\varphi : R \to X$ continuous up to the boundary the condition $\varphi(\partial R) \subset K$ imply $\varphi(R) \subset \hat{K}$.

ii) X is called disk-convex if for any convergent on $\partial\Delta$ sequence $\{\varphi_n : \bar{\Delta} \to X\}$ of analytic disks this sequence converge also on $\bar{\Delta}$.

iii) The same definition can be given with sequences of Riemann surfaces instead of the disk.

Question 6.2 What is the relation between all these notions and that defined in Definition 6.1? Are they equivalent?

Of course there are some obvious implications.

7 Open Questions

Question 7.1 Let the complex manifold D is defined as two-sheeted cover of $\Delta^2 \setminus \mathbb{R}^2$, i.e. D is a "nonschlicht" domain over $\mathbb{C}^2$. Does there exist a compact complex manifold X and a holomorphic (meromorphic) mapping $f : D \to X$ which separates points?

Note that the results of this paper imply that such X if exists cannot possed a plurinegative metric form. Thus examples could occur starting from $\mathsf{dim}\, X \geqslant 3$.

In the following problems the space X is equipped with some Hermitian metric form ω. On the subsets of $\mathbb{C}^n$ the metric is always $dd^c\|z\|^2$.

Question 7.2 Consider a class $\mathcal{J}_R$ of meromorphic mappings $f : \Delta^k \to X$, X being compact, such that

(a) $\|Df\| \geqslant R > 0$. Here $\|Df\|$ denotes the norm of the differential of f;

(b) $\mathsf{Vol}(f(\Delta^k) \leqslant C_1$ for all $f \in \mathcal{J}_R$.

Prove that there is a constant $C_2 = C_2(X, R, C_1)$ such that $\mathsf{Vol}(\Gamma_f) \leqslant C_2$ for all $f \in \mathcal{J}_R$.

To estimate the volume of the graph of f one should estimate the integral

$$\mathsf{Vol}(\Gamma_f) = \int_{\Delta^k} (dd^c||z||^2 + f^*\omega)^k = \sum_{j=0}^{k} \int_{\Delta^k} (dd^c||z||^2)^j \wedge (f^*\omega)^{k-j},$$

were only the first integral $\int_{\Delta^k}(f^*\omega)^k = \mathsf{Vol}(f(\Delta^k))$ is bounded by the condition of the question.

The following question is of the same nature.

Question 7.3 Let $f : \Delta_*^k \to X$ be a meromorphic mapping from a punctured polydisk into a compact complex space X. Suppose that $\mathsf{Vol} f(\Delta_*^k) < \infty$. Prove that f meromorphically extends to zero.

Question 7.4 Let $f : \Delta_*^{k+1} \to X \in \mathcal{G}_k$ be a meromorphic map from punctured $(k+1)$-disk into a compact complex space from class $\mathcal{G}_k$. Prove that $\mathsf{Vol}(f(A^k(r,1)) = O(\mathsf{log}^{\frac{k}{k-1}}(\frac{1}{r}))$ provided $k \geqslant 2$. In particular for equidimensional maps $f : \Delta_*^n \to X^n$ one always should have $\mathsf{Vol}(f(A^n(r,1)) = O(\mathsf{log}^{\frac{n}{n-1}}(\frac{1}{r}))$.

For $n = 1$ there are no bounds on the growth of a meromorphic function in the punctured disk.

Question 7.5 Fix some $0 < r < 1$ and some constant R. Fix also a compact complex space X. Consider the following class $\mathcal{F}_R$ of meromorphic mappings from $f : \Delta^n \to X$:

(1) $\mathsf{Vol}_{2n}(\Gamma_f \cap (A^n(r,1) \times X)) \leqslant R$;

(2) for every k-disk $\Delta_z^k = \{z\} \times \Delta^k$ (where $z \in \Delta^{n-k}$) $\mathsf{Vol}_{2k}(\Gamma_{f_z} \cap A_z^k(r,1) \times X) \leqslant R$.

Prove that for any constant l there is a constant A such that for any $f \in \mathcal{F}_R$ satisfying $\mathsf{Vol}_{2k}(\Gamma_{f_z}) \leqslant l$ for all restrictions f_z of f onto the k-disks Δ_z^k one has $\mathsf{Vol}_{2n}(\Gamma_f) \leqslant A$.

(b) Vice versa: for any constant a there is a constant L such that for any $f \in \mathcal{F}_R$ such that $\mathsf{Vol}_{2n}(\Gamma_f) \leqslant a$ one has $\mathsf{Vol}_{2k}(\Gamma_{f_s}) \leqslant L$ for all Δ_z^k.

The following question is a variation of questions 4.1 and 4.2.

Question 7.6 Let $\mathcal{X} = \{X_t\}$ be a deformation of compact complex surfaces. Suppose that X_t for $t \neq 0$ contain a global spherical shell. Does X_0 contain a GSS?

Question 7.7 Let $\mathcal{F}$ be some family of holomorphic (meromorphic) mappings from the unit polydisk Δ^{n+1} to a compact Kähler manifold X (or $X \in \mathcal{G}_1$ more generally). Suppose that $\mathcal{F}$ is equicontinuous on the Hartogs figure $H_n^1(r)$. Will $\mathcal{F}$ be equicontinuous on Δ^{n+1}?

See more about this question in [Iv3].

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